THE BITANGENTIAL OF THE QUINTIC.

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IN a paper published in the ANNALS for October, 1889, I gave a formula for the bitangential of the quintic, but in an unreduced form. I wish, first, to correct some errors in the numerical coefficients of the equation there given. In the determinant at the bottom of page 40 the factor 5 should be omitted, and in the expression for $J'(u, H, \varphi)$, page 41, the factor 18 was omitted by mistake. Making these changes, the equation of the bitangential becomes

$$[297HJ(u,H,\varphi)+456HJ'(u,H,\varphi)-20J(u,H,\theta)]^2-3025(4\theta-9H\varphi)^3=0. \quad (1)$$

It was stated at the end of the first paper, above referred to, that "The equation we have found for the bitangential contains an irrelevant factor of the eighteenth order in the variables and of the sixth in the coefficients of the given curve. This factor appears to be the square of the Hessian, and dividing out, the resulting equation is of the forty-eighth order in the variables, and of the eighteenth in the coefficients of the original equation."

It is the purpose of the present paper to show how to transform equation (1) so as to become divisible by H^2.

If we border the matrix of the Hessian both horizontally and vertically with three rows and columns, the resulting determinant is the product, with sign changed, of the two determinants added horizontally and vertically.

Thus, if F, F' be functions of the degrees n, n', we have

$$J(u,H,F)\,J(u,H,F') = \begin{vmatrix} u_1 & u_2 & u_3 \\ H_1 & H_2 & H_3 \\ F_1 & F_2 & F_3 \end{vmatrix} \times \begin{vmatrix} u_1 & u_2 & u_3 \\ H_1 & H_2 & H_3 \\ F'_1 & F'_2 & F'_3 \end{vmatrix}$$

$$= -\begin{vmatrix} a & h & g & u_1 & H_1 & F_1 \\ h & b & f & u_2 & H_2 & F_2 \\ g & f & c & u_3 & H_3 & F_3 \\ u_1 & u_2 & u_3 & & & \\ H_1 & H_2 & H_3 & & & \\ F'_1 & F'_2 & F'_3 & & & \end{vmatrix}, \quad (2)$$

where the subscripts denote differentiation with respect to x, y, z respectively.

This last determinant may be reduced by subtracting from the fourth col-

umn the sum of the first three, multiplied respectively by x, y, z; and likewise with the rows. We thus find

$$J(u,H,F)\,J(u,H,F') = -\begin{vmatrix} a & h & g & 0 & H_1 & F_1 \\ h & b & f & 0 & H_2 & F_2 \\ g & f & c & 0 & H_3 & F_3 \\ 0 & 0 & 0 & 0 & -9H & -nF \\ H_1 & H_2 & H_3 & -9H & 0 & 0 \\ F_1' & F_2' & F_3' & -n'F' & 0 & 0 \end{vmatrix}$$

$$= nn'FF'\begin{Bmatrix} H \\ H \end{Bmatrix} - 9nHF\begin{Bmatrix} F' \\ H \end{Bmatrix} - 9n'HF'\begin{Bmatrix} F \\ H \end{Bmatrix} + 81H^2\begin{Bmatrix} F \\ F' \end{Bmatrix}. \tag{3}$$

In particular,

$$[J(u,H,\theta)]^2 = 484\theta^3 - 396H\theta\begin{Bmatrix} \theta \\ H \end{Bmatrix} + 81H^2\begin{Bmatrix} \theta \\ \theta \end{Bmatrix}, \tag{4}$$

$$[J(u,H,\varphi)]^2 = 169\theta\varphi^2 - 234H\varphi\begin{Bmatrix} \varphi \\ H \end{Bmatrix} + 81H^2\begin{Bmatrix} \varphi \\ \varphi \end{Bmatrix}, \tag{5}$$

$$[J'(u,H,\varphi)]^2 = 36\theta\varphi^2 - 108H\varphi\begin{Bmatrix} \varphi' \\ H \end{Bmatrix} + 81H^2\begin{Bmatrix} \varphi' \\ \varphi' \end{Bmatrix}, \tag{6}$$

$$J(u,H,\theta)\,J(u,H,\varphi) = 286\theta^2\varphi - 117H\varphi\begin{Bmatrix} \theta \\ H \end{Bmatrix} - 198H\theta\begin{Bmatrix} \varphi \\ H \end{Bmatrix} + 81H^2\begin{Bmatrix} \theta \\ \varphi \end{Bmatrix}, \tag{7}$$

$$J(u,H,\theta)\,J'(u,H,\varphi) = 132\theta^2\varphi - 154H\varphi\begin{Bmatrix} \theta \\ H \end{Bmatrix} - 198H\theta\begin{Bmatrix} \varphi' \\ H \end{Bmatrix} + 81H^2\begin{Bmatrix} \theta \\ \varphi' \end{Bmatrix}, \tag{8}$$

$$J(u,H,\varphi)\,J'(u,H,\varphi) = 78\theta\varphi^2 - 54H\varphi\begin{Bmatrix} \varphi \\ H \end{Bmatrix} - 117H\varphi\begin{Bmatrix} \varphi' \\ H \end{Bmatrix} + 81H^2\begin{Bmatrix} \varphi \\ \varphi' \end{Bmatrix}; \tag{9}$$

where, after Clebsch, we write

$$\begin{Bmatrix} W \\ V \end{Bmatrix} = \begin{vmatrix} a & h & g & W_1 \\ h & b & f & W_2 \\ g & f & c & W_3 \\ V_1 & V_2 & V_3 & 0 \end{vmatrix},$$

and the accents denote that in performing the differentiations the second differential coefficients of H, which enter into the expression for φ, are treated as constants. Expanding equation (1) and substituting from equations (4), (5), (6), (7), (8), (9), the equation becomes

$$32400H^2\begin{Bmatrix} \theta \\ \theta \end{Bmatrix} - 1674675H^2\theta\varphi^2 + 2352240H^2\theta\begin{Bmatrix} \varphi \\ H \end{Bmatrix}$$

$$-3611520H^2\theta\begin{bmatrix}\varphi'\\H\end{bmatrix}+405000H^2\varphi\begin{bmatrix}\theta\\H\end{bmatrix}+316800H\theta^2\varphi$$
$$-158400H\theta\begin{bmatrix}\theta\\H\end{bmatrix}+2205225H^3\varphi^3-6014250H^3\varphi\begin{bmatrix}\varphi\\H\end{bmatrix}$$
$$+9234000H^3\varphi\begin{bmatrix}\varphi'\\H\end{bmatrix}-962280H^3\begin{bmatrix}\theta\\\varphi\end{bmatrix}+1477440H^3\begin{bmatrix}\theta\\\varphi'\end{bmatrix}$$
$$+7144929H^4\begin{bmatrix}\varphi\\\varphi\end{bmatrix}-21939984H^4\begin{bmatrix}\varphi\\\varphi'\end{bmatrix}+16842816H^4\begin{bmatrix}\varphi'\\\varphi'\end{bmatrix}=0. \qquad (10)$$

Every term is now divisible by H^2 except the two

$$316800H\theta^2\varphi-158400H\theta\begin{bmatrix}\theta\\H\end{bmatrix}=158400H\theta\left[2\theta\varphi-\begin{bmatrix}\theta\\H\end{bmatrix}\right].$$

To reduce this we make use of the following formula:

$$64\left[2\theta\varphi-\begin{bmatrix}\theta\\H\end{bmatrix}\right]=171H^2\psi-72H\begin{bmatrix}\varphi'\\H\end{bmatrix}+\lambda u, \qquad (11)$$

where

$$\psi=\begin{vmatrix}\frac{d^2H}{dx^2} & \frac{d^2H}{dxdy} & \frac{d^2H}{dxdz} & \frac{d}{dx}\\ \frac{d^2H}{dydx} & \frac{d^2H}{dy^2} & \frac{d^2H}{dydz} & \frac{d}{dy}\\ \frac{d^2H}{dzdx} & \frac{d^2H}{dzdy} & \frac{d^2H}{dz^2} & \frac{d}{dz}\\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} & 0\end{vmatrix}u,$$

and λ is a function of x, y, z, which for the purpose of the present paper it is not necessary to determine.

I am not in possession of an *a priori* proof of equation (11); but, as will be seen hereafter, the result to which it leads in the present problem can be verified independently. Substituting equation (11) in equation (10) and dividing by $27H^2$, we have, finally, for the equation of the bitangential,

$$1200\begin{bmatrix}\theta\\\theta\end{bmatrix}-62025\theta\varphi^2+87120\theta\begin{bmatrix}\varphi\\H\end{bmatrix}-140360\theta\begin{bmatrix}\varphi'\\H\end{bmatrix}+15000\varphi\begin{bmatrix}\theta\\H\end{bmatrix}$$
$$+H\left[\begin{matrix}81675\varphi^3-156750\psi-35640\begin{bmatrix}\theta\\\varphi\end{bmatrix}\\+54720\begin{bmatrix}\theta\\\varphi'\end{bmatrix}-222750\varphi\begin{bmatrix}\varphi\\H\end{bmatrix}+342000\varphi\begin{bmatrix}\varphi'\\H\end{bmatrix}\end{matrix}\right]$$
$$+H^2\left[264627\begin{bmatrix}\varphi\\\varphi\end{bmatrix}-812592\begin{bmatrix}\varphi\\\varphi'\end{bmatrix}+623808\begin{bmatrix}\varphi'\\\varphi'\end{bmatrix}\right]=0. \qquad (12)$$

Equation (12) can be verified in the same manner as equation (1) was verified in my first paper, and I have thought it worth while to do this for the reason that equation (11) has not been proved. Besides, this verification furnishes a check on the reductions effected in the present paper. It will be sufficient to write $c_1 = c_2 = 0$.

We have then, representing always the sum of the terms of the first and higher order in the function F by T_F,

$$H = b^2c + T_H,$$

$$\theta = 9b^6d_0^2 + 54b^5c^2d_1 + T_\theta,$$

$$\varphi = 6b^4e_0 + 24b^3cd_1 + T_\phi,$$

$$\begin{bmatrix}\theta\\ \theta\end{bmatrix} = 1296b^{14}d_0^2e_0^2 + 7776b^{13}c^2d_0e_0e_1 - 3888b^{13}cd_0^2d_1e_0 + 14256b^{13}cd_0^3e_1$$
$$+ 11664b^{12}c^4e_1^2 + 113400b^{12}c^2d_0^2d_1^2 - 36936b^{12}c^3d_0d_2e_0 + 73872b^{12}c^3d_0d_1e_1$$
$$+ 42768b^{12}c^3d_0^2e_2 - 99792b^{12}c^2d_0^3d_2 - 110808b^{11}c^5d_2e_1 + 256608b^{11}c^5d_1e_2$$
$$- 543348b^{11}c^4d_0d_1d_2 - 138996b^{11}c^4d_0^2d_3 + 662904b^{11}c^4d_1^3$$
$$+ 263169b^{10}c^6d_2^2 - 833976b^{10}c^6d_1d_3 + T_{\binom{\theta}{\theta}},$$

$$\begin{bmatrix}\varphi'\\ H\end{bmatrix} = 108b^7cd_1e_0 - 108b^7cd_0e_1 - 162b^6c^3e_2 - 1674b^6c^2d_0d_2$$
$$+ 1728b^6c^2d_1^2 - 1782b^5c^4d_3 + T_{\binom{\phi'}{H}},$$

$$\begin{bmatrix}\varphi\\ H\end{bmatrix} = 18b^8d_0f_0 + 54b^7c^2f_1 - 144b^7cd_0e_1 + 234b^7cd_1e_0 - 108b^6c^3e_2$$
$$- 2142b^6c^2d_0d_2 + 2394b^6c^2d_1^2 - 2214b^5c^4d_3 + T_{\binom{\phi}{H}},$$

$$\begin{bmatrix}\theta\\ H\end{bmatrix} = 108b^{10}d_0^2e_0 + 432b^9cd_0^2d_1 + 648b^9c^2d_0e_1 + 972b^8c^4e_2$$
$$- 3807b^8c^3d_0d_2 + 6075b^8c^3d_1^2 - 3159b^7c^5d_3 + T_{\binom{\theta}{H}},$$

$$\psi = 288b^5d_0e_1 - 288b^5d_1e_0 + 432b^4c^2e_2 - 432b^3c^3d_3$$
$$+ 576b^4cd_1^2 - 720b^4cd_0d_2 + T_\psi,$$

$$\begin{bmatrix}\theta\\ \varphi\end{bmatrix} = 1188b^{11}cd_0^2f_1 + 7128b^{10}c^3d_1f_1 + 216b^{12}d_0e_0f_0 + 648b^{11}c^2e_1f_0$$
$$- 324b^{11}cd_0d_1f_0 - 3078b^{10}c^3d_2f_0 + 1080b^{11}cd_0e_0e_1$$
$$- 2376b^{10}c^2d_0^2e_2 + 32076b^{10}cd_0^2d_1^2 - 32076b^{10}cd_0^3d_2 - 5184b^{10}c^3e_1^2$$
$$+ 8424b^{10}c^3e_0e_2 - 27864b^{10}c^2d_0d_2e_0 + 138240b^{10}c^2d_0d_1e_1 + 21762b^{10}c^2d_1^2e_0$$
$$- 48708b^9c^3d_0^2d_3 + 194400b^9c^4d_1e_2 + 279504b^9c^3d_1^3$$
$$- 258768b^9c^3d_0d_1d_2 - 27378b^9c^4d_3e_0 - 401760b^8c^5d_1d_3$$
$$+ 116964b^8c^5d_2^2 + T_{\binom{\theta}{\phi}},$$

$$\begin{bmatrix}\theta\\ \varphi'\end{bmatrix} = 3888b^{10}c^3e_0e_2 - 3888b^{10}c^3e_1^2 + 7128b^{10}c^2d_0d_1e_1 - 13608b^{10}c^2d_0d_2e_0$$
$$+ 10044b^{10}c^2d_1^2e_0 - 3564b^{10}c^2d_0^2e_2 + 28512b^{10}cd_0^2d_1^2$$
$$- 28512b^{10}cd_0^3d_2 + 4860b^9c^4d_2e_1 - 12636b^9c^4d_3e_0$$
$$- 5832b^9c^4d_1e_2 - 200556b^9c^3d_0d_1d_2 + 211248b^9c^3d_1^3$$
$$- 39204b^9c^3d_0^2d_3 + 64638b^8c^5d_2^2 - 285768b^8c^5d_1d_3 + T_{\binom{\theta}{\phi'}},$$

$$\begin{bmatrix}\varphi\\ \varphi\end{bmatrix} = 36b^{10}f_0^2 - 576b^9ce_1f_0 + 2304b^8c^2e_1^2 - 2736b^8c^2d_2f_0 + 936b^9ce_0f_1$$
$$- 1872b^8c^2e_0e_2 + 3744b^8c^2d_1f_1 + 25272b^8cd_1^2e_0 - 25272b^8cd_0d_2e_0$$
$$+ 21888b^7c^3d_2e_1 - 38376b^7c^3d_3e_0 - 7488b^7c^3d_1e_2$$
$$+ 101088b^7c^2d_1^3 - 101088b^7c^2d_0d_1d_2 + 51984b^6c^4d_2^2$$
$$- 153504b^6c^4d_1d_3 + T_{\binom{\phi}{\phi}},$$

$$\begin{bmatrix}\varphi\\ \varphi'\end{bmatrix} = 216b^9ce_0f_1 - 216b^9ce_1f_0 - 756b^8c^2d_2f_0 + 864b^8c^2d_1f_1$$
$$+ 1728b^8c^2e_1^2 - 1836b^8c^2e_0e_2 + 17064b^8cd_1^2e_0 - 17064b^8cd_0d_2e_0$$
$$- 7344b^7c^3d_1e_2 + 14256b^7c^3d_2e_1 + 68256b^7c^2d_1^3$$
$$- 68256b^7c^2d_0d_1d_2 - 24300b^7c^3d_3e_0 + 28728b^6c^4d_2^2$$
$$- 97200b^6c^4d_1d_3 + T_{\binom{\phi}{\phi'}},$$

$$\begin{bmatrix}\varphi'\\ \varphi'\end{bmatrix} = 1296b^8c^2e_1^2 - 1296b^8c^2e_0e_2 + 10368b^8cd_1^2e_0 - 10368b^8cd_0d_2e_0$$
$$- 5184b^7c^3d_1e_2 + 9072b^7c^3d_2e_1 - 14256b^7c^3d_3e_0$$
$$+ 41472b^7c^2d_1^3 - 41472b^7c^2d_0d_1d_2 + 15876b^6c^4d_2^2$$
$$- 57024b^6c^4d_1d_3 + T_{\binom{\phi'}{\phi'}}.$$

The calculation of the absolute term in equation (12) now presents no difficulty, but is extremely tedious.

I found its value to be

$$352836b^{14}(27c^2f_0^2 + 40d_0^3f_0 + 50ce_0^3 - 25d_0^2e_0^2 - 90cd_0e_0f_0),$$

which, as was shown in the former paper, vanishes if the origin is a point of contact of a bitangent. The correctness of equation (12) is thus established.